AGATES - Geometry of secants

This is a list of questions and conjectures that arose during the workshop "Geometry of Secants", held at IM PAN Warsaw in October 2022.

Conjecture 1. Given an irreducible projective variety X, there exists $d \in \mathbb{N}$ such that the d-Veronese embedding of X is not defective.

Problem 2. Classify irreducible projective varieties $X \in \mathbb{P}^N$ such that the secant variety of lines $\sigma_2(X)$ has low degree (for instance, degree 3).

For $m \in \mathbb{N}$ and $p \in \mathbb{P}^2$, define a *m-square point* supported at p as a local scheme defined by the ideal (ℓ_1^m, ℓ_2^m), where ℓ_1 and ℓ_2 are distinct lines through p.

Conjecture 3. If $Z \subset \mathbb{P}^2$ is the union of s general 2-square points, then the Hilbert function of Z is the same as the Hilbert function of $4s$ general simple points. This conjecture is true for $s \leq 7$.

Given an irreducible projective variety X and a positive integer r, the *r-Terracini locus* of X is the subset of the symmetric power $X^{(r)}$ of sets of points $\{p_1, \ldots, p_r\}$ such that the linear span $\langle \mathbb{T}_{p_1}, \ldots, \mathbb{T}_{p_r} \rangle$ of the tangent spaces has dimension smaller than the expected. One reference is [1].

Problem 4. Describe the r-Terracini loci when X is a del Pezzo surface.

Problem 5. If f is a general degree 4 form in 4 variables of rank 9, then its variety of sums of powers $\mathrm{VSP}(f, 9)$ is a degree 4 surface in $\mathbb{P}^3$ - see [2, Theorem 5.16]. Let us call D the divisor of $\mathbb{P}(\mathbb{C}[x_0, x_1, x_2, x_3])_4$ parametrizing rank 9 quaternary quartics and consider the map

$$\begin{array}{ccc} D & \dashrightarrow & \mathbb{P}(\mathbb{C}[x_0, x_1, x_2, x_3])_4 \\ f & \mapsto & \mathrm{VSP}(f, 9). \end{array}$$

What is the image of this map? Is the generic fiber zero-dimensional? If so, what is the degree of the map?

If $X \subset \mathbb{R}^N$ is a compact real manifold, the *medial axis* of X is the closure of the set of all $p \in \mathbb{R}^N$ having at least two closest points on X - see [3, Definition 2.1].

Problem 6. Classify real, compact algebraic varieties such that the medial axis is empty.

References

[1] Ballico E., Bernardi A., Santarsiero P., *Terracini locus for three points on a Segre variety*, preprint arXiv:2012.00574.

[2] Kapustka G., Kapustka M., Ranestad K., Schenck H., Stillman M., Yuan B., *Quaternary quartic forms and Gorenstein rings*, preprint arXiv:2111.05817.

[3] Di Rocco S., Edwards P., Eklund D., Gäfvert O., Hauenstein J., *Computing geometric feature sizes for algebraic manifolds*, preprint arXiv:2209.01654.